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Surface-wave generation: a viscoelastic model

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The Reynolds-averaged equations for turbulent flow over a deep-water sinusoidal gravity wave, $z = a\cos kx \equiv h_0(x)$, are formulated in the wave-following coordinates ξ, η, where $x = \xi$, $z = \eta + h(\xi, \eta)$, $h(\xi, 0) = h_0(\xi)$ and h is exponentially small for $k\eta \gg 1$, and closed by a viscoelastic constitutive equation (a mixing-length model with relaxation). This closure is derived from Townsend's boundary-layer-evolution equation on the assumptions that: the basic velocity profile is logarithmic in $\eta + z_0$, where z_0 is a roughness length determined by Charnock's similarity relation; the lateral transport of turbulent energy in the perturbed flow is negligible; the dissipation length is proportional to $\eta + z_0$. A counterpart of the Orr–Sommerfeld equation for the complex amplitude of the perturbation stream function is derived and used to construct a quadratic functional for the energy transfer to the wave. A corresponding Galerkin approximation that is based on independent variational approximations for outer (quasi-laminar) and inner (shear-stress) domains yields an energy-transfer parameter β that is comparable in magnitude with that of the quasi-laminar model (Miles 1957) and those calculated by Townsend (1972) and Gent & Taylor (1976) through numerical integration of the Reynolds-averaged equations. The calculated limiting values of β for very slow waves, with Charnock's relation replaced by kz_0 = constant, are close to those inferred from observation but about three times the limiting values obtained through extrapolation of Townsend's results.

1. Introduction

I consider here the role of wave-induced Reynolds stresses in the transfer of energy from a turbulent shear flow to gravity waves on deep water, building on the earlier investigations of Miles (1957, hereinafter referred to as M57; and 1967), Davis (1972), Townsend (1972), Belcher & Hunt (1993), and Miles (1993, hereinafter referred to as M93).† In M57, I constructed a quasi-laminar model in which turbulence is implicitly included through the prescribed velocity profile but the wave-induced Reynolds stresses are neglected. In M93, following Knight (1977), Jacobs (1987) and van Duin & Janssen (1992), I included these Reynolds stresses through an eddy-viscosity closure and the ancillary hypothesis that the eddy viscosity is conserved along streamlines. But Townsend (1972), Zeman & Jensen (1987, who also cite Taylor 1980) and Belcher & Hunt (1993) argue, and I agree, that, although an eddy-viscosity model may be appropriate for $kz \ll 1$ (k = wavenumber and z = elevation), a rapid-distortion model is more appropriate for $kz = O(1)$; indeed, the quasi-laminar model may be regarded as a limiting case of a rapid-distortion model (see comments following (1.13)).

† See Komen *et al.* (1994) for an extensive survey of theoretical, empirical and observational aspects of the wave-generation problem.

Each of the papers cited above and the present investigation consider an incompressible, inviscid, turbulent boundary-layer flow over a gravity wave of the form

$$z = a\cos kx \equiv h_0(x), \tag{1.1}$$

where x and z are Cartesian coordinates in a reference frame moving in the x-direction with the wave speed $c = (g/k)^{1/2}$. The basic velocity in this reference frame is

$$\mathscr{U} \equiv U(z) - c, \tag{1.2a}$$

where $U(z)$ typically is approximated by

$$U(z) = U_1 \ln[(z+z_0)/z_0], \quad U_1 = U_*/\kappa \tag{1.2b, c}$$

(but U and $\mathscr{U}$ ultimately are posed as functions of the wave-following coordinate η), U_*^2 is the kinematic shear stress, $\kappa \approx 0.4$ is von Kármán's constant, and z_0 is the roughness length. Note that, although (1.2b) implies $U(z) \to U_1 z/z_0$ as $z/z_0 \downarrow 0$, it is not intended to provide an adequate description of a laminar sublayer.

It is conventional (Komen *et al.* 1994, II. 2.3) to determine z_0 from Charnock's (1955) similarity relation, $gz_0/U_1^2 =$ constant, but Townsend (1972), Gent & Taylor (1976), and Belcher & Hunt (1993) assume $kz_0 = gz_0/c^2 =$ constant. On dimensional grounds alone, $gz_0/U_1^2 =$ constant and $gz_0/c^2 =$ constant are equally plausible, each being a special case of $gz_0/U_1^2 = f(c/U_1)$; however, $gz_0/U_1^2 \approx$ constant is supported by dynamical considerations for aerodynamically rough flow over wind-generated gravity waves (Phillips 1977, §4.10).

In the quasi-laminar model of M57, the energy transfer to the surface wave is associated with a singularity at the critical layer ($z = z_c$, where $\mathscr{U} = 0$) and is described by

$$\sigma \equiv (kc\bar{E})^{-1}(\partial\bar{E}/\partial t) = s\beta(U_1/c)^2, \quad \beta = -\pi(U''/kU')_c(\overline{w_c^2}/U_1^2\overline{h_{0x}^2}) \equiv \beta_c, \tag{1.3a, b}$$

where E is the wave energy, the overbar signifies an average over x, $s \equiv \rho_a/\rho_w$ is the air–water density ratio, U_1 is the reference velocity (1.2c), $' \equiv \mathrm{d}/\mathrm{d}z$, w is the wave-induced vertical velocity, and the subscript c implies $z = z_c$ ($\overline{h_0^2}$ in M93 (1.4b) should be $\overline{h_{0x}^2}$). A necessary condition for this critical-layer mechanism to be significant is $kz_c \ll 1$.

The incorporation of the wave-induced Reynolds stresses through an eddy-viscosity model and the neglect of the diffusion of the perturbation vorticity ω yields (M93)

$$\nabla\cdot(\mathscr{U}^2\nabla\zeta) + 2\kappa U_* U'\partial_x\zeta = 0 \quad (' \equiv \mathrm{d}/\mathrm{d}z) \tag{1.4}$$

for the determination of the streamline displacement ζ. This leads to $\beta = \beta_c + \beta_\nu$ in (1.3a), where β_c is given by (1.3b),

$$\beta_\nu = 2\kappa^2\mathscr{U}_o/U_1, \quad \mathscr{U}_o \equiv \mathscr{U}(z_o), \quad kz_o = \tfrac{1}{2}\mathrm{e}^{-\gamma} = 0.281 \quad (\gamma = 0.577\cdots), \tag{1.5a–c}$$

and z_o ($\equiv z_1$ in M93) is an outer length scale. The result (1.5a) is due essentially to Knight (1977), but he assumes $z_o = 1/k$, as also do Jacobs (1987) and van Duin & Janssen (1992).

The analyses in these last three papers and M93 are asymptotic for $\epsilon \downarrow 0$, where

$$1/\epsilon \equiv \mathscr{U}_o/U_1 = \ln[z_o/(z_c+z_0)] \equiv L_o. \tag{1.6}$$

Setting $U = c$ in (1.2b) to determine z_c and invoking Charnock's relation (see above) to obtain

$$\not{k} \equiv k(z_c+z_0) = C\hat{c}^{-2}\mathrm{e}^{\hat{c}}, \quad C \equiv gz_0/U_1^2, \quad \hat{c} \equiv c/U_1, \tag{1.7a–c}$$

we find that ϵ has the minimum value $1/[\ln(2/C) - 2 - \gamma]$. This minimum coincides with that of $\not{k}$, occurs at $\hat{c} = 2$, and is 0.20/0.37 for $C = 10^{-3}/10^{-2}$ (the typical range of C). Its existence implies that the limit $\epsilon \downarrow 0$ with C fixed is inaccessible. If (1.7b) is replaced by $kz_0 =$ constant, ϵ decreases monotonically with $\hat{c}$ to $-1/(\gamma + \ln 2kz_0)$ for $\hat{c} = 0$.

Belcher & Hunt (1993) assume $\hat{c} \ll 1$, neglect the wave-induced Reynolds stresses in an outer domain in which the length scale is h_m (cf. z_0), posit a mixing-length model in an inner domain in which the length scale is l (their model implies a discontinuity in the wave-induced Reynolds stresses between the outer and inner domains), and obtain

$$\beta = 2\kappa^2[2V^4 + V^2 - 1 + O(\Delta)], \quad V \equiv \mathscr{U}(h_m)/\mathscr{U}(l), \quad \Delta \equiv U_1/\mathscr{U}(l), \tag{1.8a–c}$$

where

$$(kh_m)^2 \ln(h_m/z_c) = 1, \quad kl \ln(l/z_c) = 2\kappa^2 \quad (kz_c \ll kl \ll 1). \tag{1.9a, b}$$

Their analysis is asymptotic for $\Delta \downarrow 0$, but they find that the numerical values of Δ in the physical domain of interest are as large as $\frac{1}{3}$; they also determine the $O(\Delta)$ component of (1.8a). The approximation (1.5a) is formally an order of magnitude larger than (1.8a) in the limit $\Delta \downarrow 0$, but (1.8a) may be numerically larger than (1.5a) in their range of Δ.

In the following development, I allow for a continuous transition from the quasi-laminar model in an outer domain to an eddy-viscosity model near the interface by adopting a viscoelastic closure (cf. Crow 1968; Davis 1972; Townsend 1972; Manton 1972; Abrams & Hanratty 1985). I proceed as follows. In §2, I develop the Reynolds-averaged Euler equations that govern two-dimensional wave motion on the assumptions that $\partial_x\langle u'^2 - w'^2\rangle$ and $\partial_x\langle u'w'\rangle$ are dominated by $\partial_z\langle u'w'\rangle$, where u' and w' are the x- and z-components of the velocity fluctuation and $\langle\ \rangle$ implies a y-average. I then assume steady flow in the reference frame of the sinusoidal wave (1.1), invoke $ka \ll 1$, introduce the wave-following coordinates ξ and η through the transformation

$$x = \xi, \quad z = \eta + h(\xi, \eta), \tag{1.10a, b}$$

where $h = h(\xi, \eta)$ maps $z = h_0$ on $\eta = 0$ and is evanescent for $k\eta \uparrow \infty$ but is otherwise arbitrary, and posit the solution in ξ, η as a small perturbation with respect to the basic flow $U(\eta)$.

In §3, I invoke the viscoelastic closure (Miles 1996)

$$T(\mathrm{D}\tau/\mathrm{D}t) + \tau - \tau_0 = \nu_1(\langle u_z\rangle - U'), \quad U' = U'(\eta), \tag{1.11a, b}$$

where τ is the mean shear stress, T is a relaxation time, $\tau_0 = U_*^2$ is the mean shear stress in the basic flow, ν_1 is an eddy viscosity for the wave-induced flow, and $\langle u_z\rangle$ is the mean shear-strain rate. Similarity considerations suggest $T \propto 1/U'$ and $\nu_1 \propto \nu_0 \equiv \tau_0/U'$, and I infer $T = 1/\kappa^2 U'$ and $\nu_1 = 2\nu_0$ from Townsend's (1976, §7.13) phenomenological equation for the transport of turbulent energy in a boundary layer. The introduction of the complex carrier $\exp(\mathrm{i}k\xi)$, for which $\mathrm{D}/\mathrm{D}t \rightarrow \mathrm{i}k\mathscr{U}(\eta)$, then yields the complex viscosity

$$\nu_k = \frac{2\nu_0}{1 + \mathrm{i}kT\mathscr{U}} = \frac{2U_*^2}{U'(\eta) + (\mathrm{i}k/\kappa^2)\mathscr{U}(\eta)} = \nu_k(\eta) \tag{1.12}$$

for the ratio of the complex amplitudes of $\tau - \tau_0$ and $\langle u_z\rangle - U'(\eta)$. This leads to the reduction of the Reynolds-averaged equations to the Orr–Sommerfeld-like equation

$$[\nu_k(\Phi'' + U''\mathscr{H})]'' = \mathrm{i}k[\mathscr{U}(\Phi'' - k^2\Phi) - U''\Phi] \quad (' \equiv \mathrm{d}/\mathrm{d}\eta), \tag{1.13}$$

where $\mathscr{H}(\eta)$ and $\Phi(\eta)$ are the complex amplitudes of h and ϕ, a perturbation stream function. The effective, inverse Reynolds number (the small parameter in the conventional Orr–Sommerfeld equation), $\nu_k/kl^2\mathscr{U}$ for $\eta = O(l)$, increases from $O(\epsilon^2\kappa^2)$

in the outer domain, where $l = 1/k$, to $O(1)$ in the shear-stress layer, where $l = \delta/k$ and δ satisfies $\delta \ln(\delta/\kappa) = O(\kappa^2)$. In that neighbourhood of the interface in which the relaxation time T is small compared with the transport time $1/k\mathscr{U}$, the elastic component of ν_k is negligible and (1.12) reduces to $\nu_k = 2\nu_0$, which corresponds to a mixing-length model (cf. Belcher & Hunt 1993 and van Duin & Janssen 1992). Conversely, in that outer domain in which $T \gg 1/k\mathscr{U}$, (1.12) reduces to $\nu_k = (2U_*^2/\mathscr{U})(ik)^{-1}$, which corresponds to an elastic response of the flow or, equivalently, a rapid-distortion model; however, ν_k is negligible in this domain, and (1.13) then reduces to Rayleigh's equation, corresponding to the quasi-laminar model.

$\Phi(\eta)$ is regular at $\eta = \eta_c$, but

$$\mathscr{Z} = -\Phi/\mathscr{U}, \tag{1.14}$$

the linear approximation to the complex amplitude of the streamline displacement, is singular there. This kinematic singularity reflects the presence of closed streamlines in the perturbed flow (Phillips 1977, §4.3), for the description of which the linearized formulation is inadequate (cf. Benney & Bergeron 1969). I resolve this difficulty in Appendix A by matching a local description of the closed streamlines to the outer description of §2. It appears from this matching that the external effects of the closed streamlines in $|\eta - \eta_c| \gg (a\eta_c)^{1/2}$ are captured by the singularity at $\eta = \eta_c$.

The determination of the energy-transfer parameter β, defined as in (1.3*a*), requires the calculation of the complex amplitudes of the wave-induced pressure and shear stress at the surface. These may be calculated from the solution of (1.13), subject to the boundary conditions at $\eta = 0$ and ∞. Alternatively, β may be expressed as the quadratic functional

$$\beta = (kaU_1)^{-2}\,\mathrm{Re}\int_0^\infty \{\nu_k[\mathscr{U}\mathscr{Z}''^2 + 2U'\mathscr{Z}'\mathscr{Z}'' + U''(\mathscr{Z} - \mathscr{H})\mathscr{Z}''] + ik\,\mathscr{U}^2(\mathscr{Z}'^2 + k^2\mathscr{Z}^2)\}\,\mathrm{d}\eta, \tag{1.15}$$

which provides a Galerkin approximation for suitable approximations to $\mathscr{H}$ and $\mathscr{Z}$. I carry out the latter calculation in §4 using a trial function that comprises two exponential functions with disparate outer and inner length scales similar to those introduced by Belcher & Hunt ((1.8) above). I obtain estimates of these length scales in §§5 and 6, respectively, from variational integrals for reduced forms of (1.13) in outer and inner domains. The reader who is more interested in the formulation of the problem and the results than in the technical details of the approximate solution could omit §§4–6.

The resulting approximation to β is plotted versus c/U_1 in figure 1, as also are the corresponding approximations for a mixing-length model (for which $T = 0$ in (1.11)), the quasi-laminar model (Miles 1959*a*), and the numerical integrations of Townsend (1972) and Gent & Taylor (1976). The viscoelastic approximation is remarkably close to the quasi-laminar approximation in both magnitude and shape (variation with c). The mixing-length approximation, although similar in shape, is larger than the viscoelastic approximation by almost a factor of two and, like the eddy-viscosity approximation of M93, also is larger than the quasi-laminar approximation. Townsend's β is of comparable magnitude, but with a somewhat narrower peak at a significantly larger value of c. The present approximation yields a limiting value of β for $c/U_1 \downarrow 0$ with kz_0 fixed in reasonable agreement with the limit inferred from observation, but roughly three times Townsend's (extrapolated) limit.

The differences between the present results and those of Townsend, especially for small c/U_1, are surprising, since both his model and the present model rest on a viscoelastic closure of the linearized, Reynolds-averaged Euler equations. The most plausible reasons appear to be Townsend's resolution of the basic and wave-induced flows in Cartesian, rather than wave-following, coordinates, and possible inaccuracies in his numerical integration and/or the present Galerkin approximation. Other differences between the two formulations, including Townsend's assumption $kz_0 =$ constant rather than $gz_0/U_1^2 =$ constant and his provision for the lateral transport of turbulent energy, appear to be qualitatively insignificant.

2. Equations of motion

2.1. *Reynolds-averaged equations*

We pose the velocity field in the form

$$[u_i] \equiv [u, v, w] = [\psi_z, 0, -\psi_x] + [u_i'(x, y, z, t)] \quad (i = 1, 2, 3) \tag{2.1}$$

in the Cartesian coordinates $[x_i] \equiv \{x, y, z\}$, where ψ is the stream function for the mean ($\equiv y$-averaged) flow, the u_i satisfy the continuity equation

$$\partial_i u_i = 0 \quad (\partial_i \equiv \partial/\partial x_i), \tag{2.2}$$

repeated indices are summed over 1–3, and $[u_i']$ is a randomly fluctuating velocity.

The Reynolds-averaged Euler equations are

$$\mathscr{D}\langle u_i\rangle = -\partial_i\langle p/\rho\rangle - \partial_j\langle u_i' u_j'\rangle, \quad \mathscr{D} \equiv \mathrm{D}/\mathrm{D}t = \partial_t + \langle u_j\rangle\,\partial_j, \tag{2.3a, b}$$

where $\langle\ \rangle$ implies a y-average, $\langle u_i'\rangle \equiv 0$, p is the pressure, $\rho \equiv \rho_a$ is the density, and $-\langle u_i' u_j'\rangle$ is the Reynolds-stress tensor. (All 'stresses' in the present development are true stresses divided by ρ.) Following Townsend (1972), we rewrite the x- and z-components of (2.3a) in the form

$$\mathscr{D}\langle u\rangle = -\pi_x - \chi_x + \tau_z, \quad \mathscr{D}\langle w\rangle = -\pi_z + \tau_x, \tag{2.4a, b}$$

where

$$\pi \equiv \langle p/\rho + w'^2\rangle, \quad \chi \equiv \langle u'^2 - w'^2\rangle, \quad \tau \equiv -\langle u'w'\rangle, \tag{2.5a–c}$$

$-\pi$ is the mean vertical stress, and τ is the mean shear stress. Guided by scaling arguments and Townsend's (1972) conclusion 'that the calculated solutions are not significantly different if stresses other than [τ] are ignored', we neglect χ_x and τ_x in (2.4a, b), which then reduce to

$$\mathscr{D}\langle u\rangle = -\pi_x + \tau_z, \quad \mathscr{D}\langle w\rangle = -\pi_z. \tag{2.6a, b}$$

2.2. *Wave-following coordinates*

We now introduce the coordinates ξ and η, the wave-following function h, and the perturbation stream function $\phi + \mathscr{U}h$ through the transformation (cf. Benjamin 1959)

$$x = \xi, \quad z = \eta + h(\xi, \eta), \tag{2.7a, b}$$

and

$$\psi = \int_0^\eta \mathscr{U}(\eta)\,\mathrm{d}\eta + \mathscr{U}(\eta)\,h(\xi, \eta) + \phi(\xi, \eta), \quad \mathscr{U}(\eta) \equiv U(\eta) - c, \tag{2.8a, b}$$

where $\mathscr{U}$ is the mean velocity of the basic flow in the reference frame of the wave. The linear approximations to the gradient operator, the operator $\mathscr{D}$ (2.3b), the mean velocity, and the wave-induced perturbation ω of the mean vorticity (relative to its value U' at the elevation $z-\zeta$, where ζ is the mean vertical displacement of a particle) then are given by

$$\partial_x = \partial_\xi - h_\xi\partial_\eta, \quad \partial_z = \partial_\eta - h_\eta\partial_\eta, \quad \mathscr{D} = (\mathscr{U} + U'h + \phi_\eta)\,\partial_\xi - (\mathscr{U}h_\xi + \phi_\xi)\,\partial_\eta, \tag{2.9a–c}$$

$$\langle u\rangle = \mathscr{U}(\eta) + U'(\eta)\,h + \phi_\eta, \quad \langle w\rangle = -\phi_\xi, \tag{2.10a, b}$$

and

$$\omega \equiv \langle u_z - w_x \rangle - U'(z-\zeta) \approx \phi_{\xi\xi} + \phi_{\eta\eta} + U''(\eta)\,\zeta. \tag{2.11}$$

The elimination of π between (2.6*a*, *b*) yields (without linearization)

$$\mathscr{D}\omega = \tau_{zz}. \tag{2.12}$$

The linear approximation $\mathscr{U}\zeta_\xi = -\phi_\xi$ to the kinematic equation $\mathscr{D}\zeta = \langle w \rangle$ yields

$$\zeta = -\phi/\mathscr{U}. \tag{2.13}$$

This approximation fails near $\mathscr{U} = 0$ (unless $\phi_c = 0$), but (2.8*a*) provides a description of the closed streamlines and determines ζ in this neighbourhood. (Appendix A).

The choice of h remains open, subject to the wave-following condition $h = h_0$ on $\eta = 0$ and the requirement that the influence of the wave, and hence h, must vanish as $k\eta \uparrow \infty$. These conditions are satisfied by

$$h = h_0(\xi)\exp(-k\eta), \tag{2.14}$$

which yields a non-orthogonal counterpart of Benjamin's (1959) orthogonal coordinates (derived from a potential flow over $z = h_0$). The condition $h = h_0$ at $\eta = 0$ compensates for the large velocity gradient ($U' \rightarrow U_1/z_0$) near the interface but not for the large vorticity gradient ($U'' \rightarrow -U_1/z_0^2$), for which purpose it proves expedient to invoke the additional boundary condition $h_\eta = kh_0$ (whereas (2.14) implies $h_\eta = -kh_0$). Summing up, we require

$$h = h_0, \quad h_\eta = kh_0 \quad (\eta = 0), \quad h \rightarrow 0 \quad (k\eta \uparrow \infty), \tag{2.15a–c}$$

the satisfaction of which ultimately leads to a generalization of (2.14) that incorporates separate outer and inner length scales; see (4.7). A natural choice that satisfies (2.15*a–c*) is $h = \zeta$ (Miles 1967), which maps the streamlines of the mean flow on the lines of constant η. However, this implicitly assumes that all streamlines originate in the basic flow, thereby excluding closed streamlines, and renders the governing equations singular at the critical layer (see §3).

2.3. *Monochromatic motion*

It follows from the assumptions of monochromatic mean motion and linearity that the wave-induced perturbations with respect to the basic flow $\mathscr{U}(\eta)$ admit the representation

$$[h, \zeta, \phi, \omega, \pi, \tau - \tau_0] = \mathrm{Re}\{[\mathscr{H}, \mathscr{Z}, \Phi, \Omega, \mathscr{P}, \mathscr{T}]\,\mathrm{e}^{\mathrm{i}k\xi}\}, \tag{2.16}$$

where $\mathscr{H} \cdots$ are complex amplitudes. Combining (2.6), (2.10)–(2.12) and (2.16), we obtain the linear approximations

$$\mathscr{P} = U'\Phi - \mathscr{U}\Phi' + (\mathrm{i}k)^{-1}\mathscr{T}', \quad \mathscr{P}' = -k^2\mathscr{U}\Phi, \tag{2.17a, b}$$

and

$$\mathscr{U}\Omega = \mathscr{U}(\Phi'' - k^2\Phi) - U''\Phi = (\mathrm{i}k)^{-1}\mathscr{T}'' \quad (' \equiv \mathrm{d}/\mathrm{d}\eta). \tag{2.18}$$

Continuity of the interfacial velocity (we neglect the wind-induced drift) and evanescence of the wave-induced disturbance as $k\eta \uparrow \infty$ imply the boundary conditions

$$\Phi = ac, \quad \Phi' = a(kc - U') \quad (\eta = 0); \quad \Phi \rightarrow 0, \quad \mathscr{T} \rightarrow 0 \quad (k\eta \uparrow \infty). \tag{2.19a–d}$$

Lamb's (1932, §349) solution for a surface wave in a viscous liquid with prescribed stresses at the surface may be adapted to the present problem through the limit $s \equiv \rho_a/\rho_w \downarrow 0$ (Appendix B) to obtain the *interfacial impedance* (defined as in M57)

$$\alpha + \mathrm{i}\beta \equiv (c^2 - c_w^2)/sU_1^2 = (\mathscr{P}_0 + \mathrm{i}\mathscr{T}_0)/kaU_1^2 \equiv \hat{\mathscr{P}}_0 + \mathrm{i}\hat{\mathscr{T}}_0, \tag{2.20}$$

where c is the complex wave speed and c_w (the imaginary part of which comprehends viscous dissipation in the water) is its value in the absence of the air.

3. Viscoelastic closure

We proceed on the assumptions that: (i) the basic flow in the wave-following coordinates ξ, η is described by

$$U(\eta) = \frac{U_*}{\kappa}\ln\left(\frac{\eta+z_0}{z_0}\right), \quad \nu_0(\eta) = \frac{U_*^2}{U'(\eta)} = \kappa U_*(\eta+z_0), \tag{3.1 a, b}$$

where z_0 is the surface roughness and ν_0 is the basic eddy viscosity; (ii) the evolution of the shear stress τ in the perturbed boundary layer is governed by (Townsend 1976, §7.13)

$$\tfrac{1}{2}a_1^{-1}\mathscr{D}\tau - \tau\langle u_z\rangle + L_\epsilon^{-1}(\tau/a_1)^{3/2} = -\partial_z\langle(p'/\rho)\,w' + \tfrac{1}{2}q^2w'\rangle \equiv Z, \tag{3.2a}$$

where

$$a_1 = \tau_0/\langle q^2\rangle \tag{3.2b}$$

is an empirical constant, for which Townsend chooses 0.15–0.16 $\approx \kappa^2$, $q^2 = u_i' u_i'$ is the turbulent intensity ($a_1 = \kappa^2$ implies $\langle q^2\rangle = U_*^2/\kappa^2 \equiv U_1^2$), L_ϵ is a dissipation length, p' and w' are the fluctuations in p and w, and Z represents the lateral transport of turbulent energy.

Townsend (1972) models Z by the gradient-diffusion form $Z = \frac{1}{2}a_1^{-1}D(\nu_0\tau_z)_z$ and approximates D by 0.3 (also, in a few places, by the alternative value 0.1, with little change in his numerical results). This form is questionable (Bradshaw, Ferriss & Atwell 1967); however, both Townsend's results and those of Bradshaw *et al.* support the neglect of Z in the present context.

The crucial construct for the implementation of (3.2) is L_ϵ. Townsend (1972) argues that L_ϵ should be proportional to $z-h_0$ near the surface but 'more nearly proportional to height above the average position of the surface' for $kz = O(1)$ and posits

$$a_1^{3/2}L_\epsilon = \kappa[z-h_0(x)\,\mathrm{e}^{-kz}]. \tag{3.3}$$

We satisfy Townsend's premises (quoted above) in the wave-following coordinates ξ, η through the somewhat simpler choice (cf. Prandtl's mixing length)

$$a_1^{3/2}L_\epsilon = \kappa(\eta+z_0). \tag{3.4}$$

Substituting (3.2*b*), (3.4) and $Z=0$ into (3.2*a*), linearizing in $\tau-\tau_0$ and the perturbation strain rate $\langle u_z\rangle - U'(\eta)$, dividing the result by $\frac{1}{2}U'(\eta)$, and introducing the relaxation time

$$T \equiv 1/a_1\,U'(\eta), \tag{3.5}$$

we obtain the viscoelastic constitutive equation (Miles 1996)

$$T\mathscr{D}\tau + \tau - \tau_0 = 2\nu_0\,\epsilon, \quad \epsilon \equiv \langle u_z\rangle - U'(\eta) \approx \phi_{\eta\eta} + U''(\eta)\,h, \tag{3.6a, b}$$

which differs essentially from Townsend's (1972) equation (2.12) only in the neglect of lateral transport and choice of L_ϵ. The complex amplitude of $\tau-\tau_0$, as defined by (2.16), then is

$$\mathscr{T} = \nu_k(\Phi'' + U''\mathscr{H}) \tag{3.7a}$$

$$= -\nu_k[\mathscr{U}\mathscr{Z}'' + 2U'\mathscr{Z}' + U''(\mathscr{Z}-\mathscr{H})] \quad (\mathscr{Z} = -\Phi/\mathscr{U}), \tag{3.7b}$$

where

$$\nu_k = 2\nu_0(1+\mathrm{i}kT\mathscr{U})^{-1} = 2U_*^2[U'(\eta)+(\mathrm{i}k/a_1)\mathscr{U}(\eta)]^{-1} \tag{3.8}$$

is a complex eddy viscosity. The limit $T\downarrow 0$ yields a mixing-length model with $\nu_k = 2\nu_0$.

Substituting (3.7a) into (2.18), we obtain the Orr–Sommerfeld-like equation

$$[\nu_k(\Phi''+U''\mathscr{H})]'' = \mathrm{i}k[\mathscr{U}(\Phi''-k^2\Phi)-U''\Phi] \tag{3.9}$$

or, alternatively, from (3.7b),

$$\{\nu_k[\mathscr{U}\mathscr{Z}''+2U'\mathscr{Z}'+U''(\mathscr{Z}-\mathscr{H})]\}'' = \mathrm{i}k[(\mathscr{U}^2\mathscr{Z}')'-k^2\mathscr{U}^2\mathscr{Z}]. \tag{3.10}$$

Note that $\eta=\eta_c$ is an ordinary point for (3.9) and a removable singular point for (3.10) if $\mathscr{H}$ is prescribed. But if $\mathscr{H}=\mathscr{Z}$ (see last paragraph in §2.2) $\eta=\eta_c$ is a regular, non-removable singular point for both (3.9) and (3.10).

4. Quadratic integral for $\alpha+\mathrm{i}\beta$

We now express $\alpha+\mathrm{i}\beta$ as a quadratic functional of $\mathscr{Z}$. Multiplying (3.10) by $-\mathscr{Z}$, integrating by parts over $0<\eta<\infty$ along a path that passes under $\eta=\eta_c$, and invoking

$$\mathscr{Z}=a,\quad \mathscr{Z}'=ka,\quad \mathscr{T}'=\mathrm{i}k(\mathscr{P}_0-kac^2)\quad (\eta=0), \tag{4.1a–c}$$

which follow from (2.17a) and (2.19a, b), and null conditions for $\eta\to\infty$, we obtain

$$\int_0^\infty \mathscr{Z}\mathscr{T}''\,\mathrm{d}\eta = ka(\mathscr{T}_0-\mathrm{i}\mathscr{P}_0)+\mathrm{i}(kac)^2+\int_0^\infty \mathscr{Z}''\mathscr{T}\,\mathrm{d}\eta \tag{4.2a}$$

$$= -\mathrm{i}k\int_0^\infty \mathscr{Z}[(\mathscr{U}^2\mathscr{Z}')'-k^2\mathscr{U}^2\mathscr{Z}]\,\mathrm{d}\eta = \mathrm{i}(kac)^2+\mathrm{i}k\int_0^\infty \mathscr{U}^2(\mathscr{Z}'^2+k^2\mathscr{Z}^2)\,\mathrm{d}\eta. \tag{4.2b, c}$$

It then follows from (2.20) and (3.7b) that

$$\alpha+\mathrm{i}\beta = (kaU_1^2)^{-1}(\mathscr{P}_0+\mathrm{i}\mathscr{T}_0) \tag{4.3a}$$

$$= (kaU_1)^{-2}\int_0^\infty \{\mathrm{i}\nu_k[\mathscr{U}\mathscr{Z}''^2+2U'\mathscr{Z}'\mathscr{Z}''+U''(\mathscr{Z}-\mathscr{H})\mathscr{Z}'']-k\mathscr{U}^2(\mathscr{Z}'^2+k^2\mathscr{Z}^2)\}\,\mathrm{d}\eta. \tag{4.3b}$$

The functional (4.3b) is not a variational integral for (3.10), which is not self-adjoint, but it does provide a viable Galerkin approximation on the assumption of a suitable trial function (or set of trial functions).

Substituting ν_k from (3.8), neglecting kz_0, introducing the dimensionless variables

$$\hat\eta = k\eta,\quad L(\hat\eta)=\mathscr{U}/U_1\approx \ln(\hat\eta/\hat k),\quad D(\hat\eta)=1+\mathrm{i}(\hat\eta/a_1)L(\hat\eta),\quad \hat{\mathscr{H}}(\hat\eta)=\mathscr{H}/a,\quad \hat{\mathscr{Z}}(\hat\eta)=\mathscr{Z}/a, \tag{4.4a–e}$$

in (4.3b), and then dropping the hats, we obtain

$$\alpha+\mathrm{i}\beta = 2\mathrm{i}\kappa^2(I+J+K)-M, \tag{4.5}$$

where

$$I=\int_0^\infty \frac{\eta L\mathscr{Z}''^2\,\mathrm{d}\eta}{D},\quad J=2\int_0^\infty \frac{\mathscr{Z}'\mathscr{Z}''\,\mathrm{d}\eta}{D},\quad K=\int_0^\infty \frac{(\mathscr{H}-\mathscr{Z})\mathscr{Z}''\,\mathrm{d}\eta}{\eta D}, \tag{4.6a–c}$$

$$M=\int_0^\infty L^2(\mathscr{Z}'^2+\mathscr{Z}^2)\,\mathrm{d}\eta, \tag{4.6d}$$

and the primes now signify differentiation with respect to the dimensionless arguments. It follows from $\mathscr{H}=\mathscr{Z}=1$ at $\eta=0$ that the integrand in (4.6c) is bounded for $\eta\to 0$.

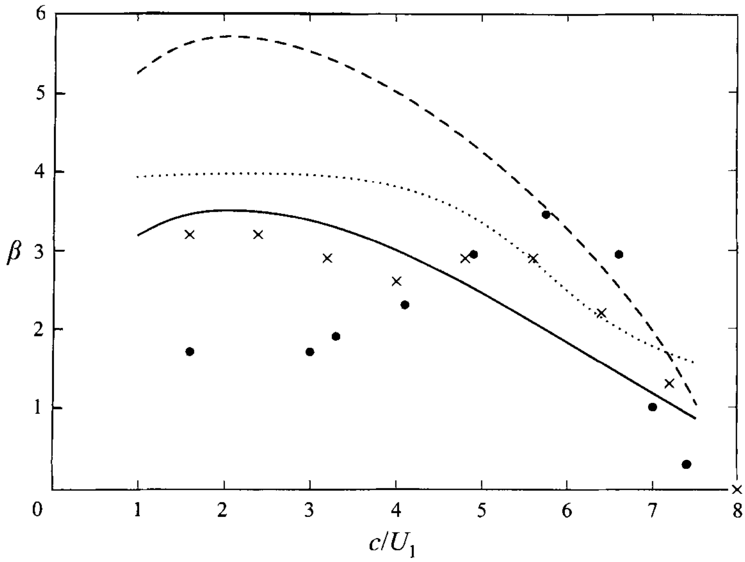

FIGURE 1. The energy-transfer parameter β, as calculated from: the viscoelastic approximation, (4.5)–(4.9) for $\kappa = 0.4$ and $C \equiv gz_0/U_1^2 = 2.3\times10^{-3}$ (——); the eddy-viscosity approximation, (4.5)–(4.8) and (4.10) with $D = 1$ (– – –); the quasi-laminar approximation (5.7*b*) (· · · ·); Townsend's (1972) numerical integration for $R \equiv \ln(1/kz_0) = 8$ (●); Gent & Taylor's (1976) numerical integration for $R = 8$ (×) and $ka = 0.01$. The assumption of constant gz_0/U_1^2 is expected to fail for small c/U_1, and the assumption $\epsilon \ll 1$ fails for $c/U_1 \gtrsim 6$.

A suitable trial function, which satisfies the dimensionless counterparts of (4.1*a*, *b*), the corresponding conditions for $\mathscr{H}$, and the null conditions at $\eta = \infty$, is

$$\mathscr{H} = \mathscr{L} = \frac{\lambda(1+\delta)\,e^{-\eta/\lambda} - \delta(1+\lambda)\,e^{-\eta/\delta}}{\lambda-\delta}, \tag{4.7}$$

where $\lambda = O(1)$ and $|\delta| \ll 1$ (δ may be complex) are outer and inner scaling parameters (see below). The resulting I and J are analytically intractable (but see Appendix C) except in the limit $D \rightarrow 1$; however, their numerical evaluation, along with that of M, is straightforward. $K \equiv 0$ by virtue of $\mathscr{H} = \mathscr{L}$.

Plausible estimates of λ and δ are provided by variational formulations of the quasi-laminar (§5) and shear-stress-layer (§6) problems, which yield

$$\lambda = \frac{1-\epsilon}{1+\epsilon} + O(\epsilon^3) \tag{4.8}$$

and

$$i\delta L_\delta/\kappa^2 = \sqrt{3}-1+O(\delta), \quad L_\delta \equiv L_o + \ln\delta, \quad L_o \equiv 1/\epsilon = -\gamma - \ln 2\ell. \tag{4.9a–c}$$

The numerical solution of (4.9*a*) for $\kappa = 0.4$ and $C = 2.3\times10^{-3}$ yields $\delta_r/\epsilon \approx 0.18$ and $\delta_i/\epsilon = -0.23$ for $0.5 < c/U_1 < 5$.

The end result for β, obtained by combining (4.5)–(4.9), (1.6) and (1.7) with $\kappa = 0.4$, $a_1 = \kappa^2 = 0.16$, and $C = gz_0/U_1^2 = 2.3\times10^{-3}$, is plotted in figure 1. Also plotted are: the corresponding approximation for a mixing-length model, for which $\nu_k = 2\nu_0$, $D = 1$, (4.9*a*) is replaced by (see §6)

$$i\delta L_\delta/\kappa^2 = 1+O(\delta), \tag{4.10}$$

and the integrals in (4.6) admit analytical evaluation (Appendix C); the quasi-laminar approximation β_c (5.7); Townsend's (1972) results for $R \equiv -\ln kz_0 = 8$; Gent & Taylor's (1976) results for $ka = 0.01$ and $R = 8$.

5. Quasi-laminar approximation

We now return to dimensional variables and approximate (2.17 *a, b*) and (3.10) in the quasi-laminar domain by

$$\mathscr{P} = U'\Phi - \mathscr{U}\Phi' = \mathscr{U}^2\mathscr{X}', \quad \mathscr{P}' = -k^2\mathscr{U}\Phi = k^2\mathscr{U}^2\mathscr{X} \tag{5.1 a, b}$$

and

$$(\mathscr{U}^2\mathscr{X}')' - k^2\mathscr{U}^2\mathscr{X} = 0. \tag{5.2}$$

Multiplying (5.2) by $\mathscr{X}$, integrating by parts over $0 < \eta < \infty$, and invoking the inner limits $\mathscr{X} \to a$ and $\mathscr{U}^2\mathscr{X}' \to \mathscr{P}_0$ and a null condition at $\eta = \infty$, we obtain (cf. Miles 1959 *b*)

$$a\mathscr{P}_0 = -\int_0^\infty \mathscr{U}^2(\mathscr{X}'^2 + k^2\mathscr{X}^2)\,d\eta, \tag{5.3}$$

which is stationary with respect to first-order variations of $\mathscr{X}$ about the true solution of (5.2).

Perhaps the simplest admissible trial function for the variational integral (5.3) is (cf. (4.7))

$$\mathscr{X} = a\,e^{-k\eta/\lambda}, \tag{5.4}$$

where λ is a free parameter. Combining (5.4) and $\mathscr{U} \approx U_1 \ln(\eta/\eta_c)$ in (5.3) yields

$$\hat{\mathscr{P}}_0 \equiv \mathscr{P}_0/kaU_1^2 = -k(\lambda^{-2}+1)\int_0^\infty e^{-2k\eta/\lambda}\ln^2(\eta/\eta_c)\,d\eta = -\tfrac{1}{2}(\lambda+\lambda^{-1})(L_\lambda^2 + \tfrac{1}{6}\pi^2), \tag{5.5}$$

where $L_\lambda \equiv L_0 + \ln\lambda$ (as in §4), and L_0 is defined as in (1.6), but with z_0 neglected. It then follows from the variational condition $\partial\hat{\mathscr{P}}_0/\partial\lambda = 0$ that

$$\lambda^2 = \frac{L_\lambda^2 - 2L_\lambda + \frac{1}{6}\pi^2}{L_\lambda^2 + 2L_\lambda + \frac{1}{6}\pi^2}, \tag{5.6}$$

the expansion of which in powers of $\epsilon \equiv 1/L_0$ yields (4.8).

The corresponding quasi-laminar approximation to the energy-transfer parameter may be calculated from (5.1 *a*), which implies $\Phi_c = \mathscr{P}_c/U_c' \approx \mathscr{P}_0/U_c'$, and (1.3 *b*), which yields

$$\beta_c = \pi\mathscr{k}^{-1}\,|\Phi_c/U_1 a|^2 = \pi\mathscr{k}\,|\hat{\mathscr{P}}_0|^2 = \tfrac{1}{4}\pi\mathscr{k}(\lambda+\lambda^{-1})^2(L_\lambda^2+\tfrac{1}{6}\pi^2)^2 \tag{5.7 a}$$

$$= \pi\mathscr{k}L_0^4[1 - (4 - \tfrac{1}{3}\pi^2)\,\epsilon^2 + O(\epsilon^3)]. \tag{5.7 b}$$

6. Shear-stress layer

We now assume $k\eta \ll 1$, neglect $k^2\mathscr{U}^2\mathscr{X}$, choose $\mathscr{H} = \mathscr{X}$ in (3.7 *b*) and (3.9), and combine the results to obtain the shear-stress-layer approximation

$$\mathscr{T}'' - (ik\mathscr{U}/\nu_k)\mathscr{T} = 0 \quad (k\eta \ll 1, \quad \mathscr{H} = \mathscr{X}). \tag{6.1}$$

Multiplying (6.1) by $\mathscr{T}$ and integrating by parts over $0 < \eta < \infty$ yields (cf. (5.3))

$$(\mathscr{T}\mathscr{T}')_0 = -\int_0^\infty [\mathscr{T}'^2 + (ik\mathscr{U}/\nu_k)\mathscr{T}^2]\,d\eta, \tag{6.2}$$

which is stationary with respect to first-order variations of $\mathscr{T}$ about the true solution of (6.1).

Proceeding as in §5, we substitute the trial function

$$\mathscr{T} = \mathscr{T}_0 e^{-k\eta/\delta} \tag{6.3}$$

and ν_k (3.8), with $a_1 = \kappa^2$ therein, into (6.2) and invoke $\partial(\mathscr{T}_0'/\mathscr{T}_0)/\partial\delta = 0$ to obtain

$$\mathscr{S} \equiv -\mathscr{T}_0'/ik\mathscr{T}_0 = \tfrac{1}{4}\kappa^{-2}[(1+\delta_*)(L_\delta^2+\tfrac{1}{6}\pi^2)+2\delta_*^{-1}-\hat{c}^2], \tag{6.4}$$

$$(L_\delta^2+\tfrac{1}{6}\pi^2+2L_\delta)\,\delta_*^2+2L_\delta\delta_*-2=0, \quad L_\delta \equiv L_o+\ln\delta, \quad \delta_* \equiv i\delta/\kappa^2, \tag{6.5a–c}$$

where $L_o \equiv 1/\epsilon$ is defined by (1.6), and $\hat{c} \equiv c/U_1$. Solving (6.5a) as a quadratic in $\delta_* L_\delta$ and letting $\delta_* \to 0$, we obtain (4.9a). The numerical solution of (6.5) for $\kappa = 0.4$ and $C = 2.3\times 10^{-3}$ yields $\delta_r/\epsilon = 0.13$ and $\delta_i/\epsilon = -0.20$ for $0.5 < \hat{c} < 5$, which compare with 0.18 and -0.23 for (4.9). The corresponding approximations to the real and imaginary parts of $\mathscr{S}$ are within 1 % and 10 %, respectively, of those obtained through numerical integration of (6.1) for $0.5 < \hat{c} < 5$.

Repeating the derivation of (6.4) and (6.5) with $\nu_k = 2\nu_0$ (the mixing-length model, for which relaxation is neglected), and letting $\delta_* \to 0$, we obtain (4.10).

7. The limit $\hat{c}\downarrow 0$

We replace Charnock's relation $gz_0/U_1^2 = C$ by Townsend's (1972)

$$kz_0 = gz_0/c^2 = \text{constant} \equiv e^{-R} \tag{7.1}$$

in the limit $\hat{c}\downarrow 0$. This implies the replacement of (1.6) and (1.7a) by

$$L_o \equiv 1/\epsilon = R-\hat{c}-1.27, \quad \hbar = e^{\hat{c}-R}. \tag{7.2a, b}$$

The corresponding, limiting values of β given by (C 8b) and (C 10b), respectively, are

$$\beta \to 0.88R-1.88+O(\epsilon) \quad (\hat{c}\downarrow 0) \tag{7.3}$$

and

$$\beta \to 1.24R-1.54+O(\epsilon) \quad (T=0, \hat{c}\downarrow 0), \tag{7.4}$$

which yield $\beta = 5.16$ and 8.37 for $R = 8$. Townsend's (1972) extrapolated limits of β for $R = 8$, 10 and 12 ($R = 10$ and 12 appear to be reversed in the caption to his figure 1) are 1.8, 1.4 and 1.2, which, in contrast to (7.3) and (7.4), are decreasing in R.

The corresponding limit implied by Belcher & Hunt's (1993) result, (1.8) above, is

$$\beta \to 0.32(2V^4+V^2-1), \quad V = \frac{R+\ln\hbar}{R+\ln l}, \tag{7.5a, b}$$

where $\hbar \equiv kh_m$ and $\ell \equiv kl$ are determined by

$$\hbar^2(R+\ln\hbar) = 1, \quad \ell(R+\ln l) = 0.32. \tag{7.5c, d}$$

Letting $R = 8$, we obtain $\hbar = 0.38$, $\ell = 0.061$, $V = 1.32$ and $\beta = 2.38$.

The corresponding values of β_ν and β_c, are, from (1.5), (5.7) and (7.2), 2.15 and 2.16.

The asymptotic value of β inferred from (1.3a), using Plant's (1982) data for the dimensionless growth rate σ, is $\beta = 5.0$. This is close to the 5.16 given by (7.3) for $R = 8$, but the uncertainty in R is too large to infer agreement to better than a factor of two.

8. Conclusion

We conclude that the difference in the wind-to-wave energy transfer predicted by the quasi-laminar and viscoelastic models is small over a wide range of c/U_1 for a logarithmic mean-wind profile with the roughness length z_0 determined by Charnock's similarity relation ($gz_0/U_1^2 =$ constant). The difference between the predictions of the quasi-laminar and eddy-viscosity models is rather larger, and in the opposite direction, but still within a factor of two. To the extent that these results are used to guide the construction of *ad hoc* approximations to the energy-transfer parameter β (Komen *et al.* 1994), their immediate practical value may lie primarily in providing confidence that the predictions of the theoretical models are relatively insensitive to significant changes in the closure hypotheses for these models. In addition, and perhaps surprisingly, they demonstrate that the established effectiveness of Townsend's boundary-layer-evolution equation in dealing with turbulent flow over fixed boundaries carries over to flow over progressive water waves.

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Appendix A. Closed streamlines

Closed streamlines occur in the neighbourhood of $\eta = \eta_c$, where $\mathscr{U}(\eta) = 0$ and the stream function ψ_b for the basic flow has the minimum (for $U_c' > 0$)

$$\psi_b \equiv \int_0^\eta \mathscr{U}(\eta)\,\mathrm{d}\eta \to \psi_c + \tfrac{1}{2}U_c'(\eta-\eta_c)^2 \quad (\eta \to \eta_c), \quad \psi_c \equiv \int_0^{\eta_c} \mathscr{U}(\eta)\,\mathrm{d}\eta \quad \text{(A 1}a, b)$$

($\psi_c \approx -U_c'\eta_c^2$ for $\mathscr{U} = U_1 \log(\eta/\eta_c)$). The stream function for the perturbed flow, (2.8*a*), admits the corresponding expansion (cf. Lighthill 1962 and Phillips 1977, §4.3)

$$\psi = \psi_c + \tfrac{1}{2}U_c'(\eta-\eta_c)^2 + U_c'(\eta-\eta_c)\,h_c(\xi) + \phi_c(\xi), \quad \text{(A 2)}$$

wherein the subscript c implies $\eta = \eta_c$, and an error factor of $1 + O(ka)$ is implicit. Introducing

$$\phi_c = \mathrm{Re}\{\Phi_c \mathrm{e}^{\mathrm{i}k\xi}\} \equiv -\tfrac{1}{4}\mathscr{A}^2 U_c' \cos k(\xi - x_0), \quad \psi_0 \equiv \psi_c - \tfrac{1}{4}\mathscr{A}^2 U_c', \quad \psi_1 = \psi_c + \tfrac{1}{4}\mathscr{A}^2 U_c', \quad \text{(A 3}a\text{–}c)$$

where

$$\Phi_c \equiv \tfrac{1}{4}\mathscr{A}^2 U_c' \, \mathrm{e}^{\mathrm{i}(\pi - kx_0)} \quad \text{(A 4)}$$

is determined by the outer solution, we transform (A 2) to

$$(\eta - \eta_c + h_c)^2 + \mathscr{A}^2 \sin^2[\tfrac{1}{2}k(\xi - x_0)] = h_c^2 + (2/U_c')(\psi - \psi_0) \quad (\psi_0 \leqslant \psi \leqslant \psi_1). \quad \text{(A 5)}$$

Equation (A 5) describes a periodic sequence of nested sets of closed streamlines (on which $z = \eta + h \approx \eta + h_c$, so that the displacement of a particle from its ambient position $z = \eta_c$ is $\zeta = \eta - \eta_c + h_c$) with the family parameter ψ. If $h_c^2 \ll \mathscr{A}^2$, as is typically true and we assume for algebraic simplicity: the centres are at $k(\xi - x_0) = 0 \bmod 2\pi$ and $\eta = \eta_c$, where $\psi = \psi_0$; the separatrix, outside of which the streamlines are sinuous, is at $\psi = \psi_1$, and the maximum thickness of the separatrix is $2\mathscr{A}$. We infer from the development of §§2–5 that $\mathscr{A} = O[(ka)^{1/2}\eta_c/\delta]$. These closed streamlines bifurcate from the minimum of ψ_b, and the entire set may be regarded as descending from the basic streamline $z = \eta = \eta_c$, while the sinuous streamlines above/below the separatrix may be regarded as descending from the basic streamlines $z = \eta \gtrless \eta_c$ (but note that $\psi > \psi_1$ on all sinuous streamlines).

Appendix B. The interfacial impedance

The solution of the linearized Navier–Stokes equations in the semi-infinite body of water bounded above by the surface wave

$$z = a\, e^{ik(x-ct)} \equiv h_0 \tag{B 1}$$

in a *fixed* reference frame yields (Lamb 1932, §349 after letting $A = iac(1+\mathscr{C})$, $C = ac\mathscr{C}$, $n = -ikc$, and neglecting surface tension therein; following Lamb, but contrasting with the notation in §§1–6 above, we now work with complex dependent variables)

$$u_w = [k+\mathscr{C}(k-m)]\,ch_0, \quad \tau_w \equiv (\tau_{13})_w = (2\nu_w kc + i\mathscr{C}c^2)\,kh_0, \tag{B 2a, b}$$

and

$$\left(-\frac{p}{\rho}+\tau_{33}\right)_w = \left\{\frac{g}{k}-c^2-2ik\nu_w c-\mathscr{C}[c^2+2i(k-m)\,\nu_w c]\right\}kh_0 \tag{B 2c}$$

for the tangential velocity, tangential stress, and normal stress, respectively, at the surface. The subscript w refers to the water, and $m \equiv [k^2 - i(kc/\nu_w)]^{1/2}$.

Invoking continuity of the perturbation velocity $u = ch_\eta$ and τ and $-\pi$, the tangential and normal stresses, eliminating $\mathscr{C}$, and letting $k\nu_w/c \downarrow 0$, we obtain the interfacial conditions

$$u - i(kc\nu_w)^{-1/2} s\tau = kch_0, \quad s(\pi + i\tau) = (c^2 - c_w^2)\,kh_0 \quad (s \equiv \rho_a/\rho_w), \tag{B 3a, b}$$

where

$$c_w \equiv (g/k)^{1/2} - 2ik\nu_w \quad (|k\nu_w/c| \ll 1) \tag{B 4}$$

is the complex wave speed in the absence of the air and comprehends (through its imaginary part, which may be replaced by an empirical equivalent) the dissipation in the water. The ratio of the second term to the first term on the left-hand side of (B 3a) is typically smaller than 10^{-2}; accordingly, (B 3a) may be approximated by $u = kch_0$, as anticipated in (2.19b). But note that (B 3a) does *not* reduce to $c = kch_0$ in the limit of an inviscid liquid ($k\nu_w/c \to 0$).

Finally, we replace π and τ by their complex amplitudes $\mathscr{P}$ and $\mathscr{T}$ (2.15) to obtain the *interfacial impedance* (defined as in M57) in the form

$$\alpha + i\beta \equiv \frac{c^2 - c_w^2}{sU_1^2} = \frac{(\mathscr{P}+i\mathscr{T})_0}{kaU_1^2} \equiv \hat{\mathscr{P}}_0 + i\hat{\mathscr{T}}_0. \tag{B 5a, b}$$

Appendix C. Approximate evaluation of I and J

The integrals I and J (4.6a, b) are analytically intractable except in the limit $D \to 1$, but reference to a table of Laplace transforms of logarithmic functions suggests that we approximate them by separating out the limits for $D \to 1$ and then letting $L(\eta) = L(\frac{1}{2}a)$ in that component of the remainder that is dominated by $\exp(-2\eta/a)$, where a is (for the moment) a free parameter; e.g. (η now is dimensionless, as in §4),

$$\int_0^\infty \frac{e^{-2\eta/a}}{D(\eta)}\,d\eta \approx \int_0^\infty \frac{e^{-2\eta/a}\,d\eta}{1+i(L_a/a_1)\eta} = \tfrac{1}{2}a\mathscr{E}_a \tag{C 1}$$

and

$$\int_0^\infty \frac{e^{-2\eta/a}\eta\, L(\eta)\,d\eta}{D(\eta)} = \tfrac{1}{4}a^2 + \tfrac{1}{2}a(a_1/i)(1-\mathscr{E}_a), \tag{C 2}$$

where

$$\mathscr{E}_a \equiv \mathscr{E}(2a_1/iaL_a), \quad L_a \equiv -\gamma - \ln(2k/a) = L_o + \ln a, \tag{C 3}$$

and

$$\mathscr{E}(z) \equiv z e^z \int_z^\infty t^{-1} e^{-t}\,dt = z e^z E_1(z). \tag{C 4}$$

(Note that $D \to 1$ corresponds to $\mathscr{E} \sim 1 - z^{-1}$.) The end results of these approximations, together with the exact evaluation of M (4.6d), are

$$I = \frac{a_1}{2i}\left[\frac{A^2}{\lambda}(1-\mathscr{E}_\lambda) + \frac{B^2}{\delta}(1-\mathscr{E}_\delta) - \frac{4AB}{\lambda+\delta}(1-\mathscr{E}_{\hat{\delta}})\right] + \tfrac{1}{2}AB\left(\frac{\lambda-\delta}{\lambda+\delta}\right)^2, \tag{C 5a}$$

$$J = -(A^2\mathscr{E}_\lambda + B^2\mathscr{E}_\delta - 2AB\mathscr{E}_{\hat{\delta}}), \tag{C 5b}$$

and

$$M = \tfrac{1}{2}A^2\lambda(1+\lambda^2)(L_\lambda^2 + \tfrac{1}{6}\pi^2) + \tfrac{1}{2}B^2\delta(1+\delta^2)(L_\delta^2 + \tfrac{1}{6}\pi^2) - AB\hat{\delta}(1+\delta\lambda)(L_{\hat{\delta}}^2 + \tfrac{1}{6}\pi^2), \tag{C 5c}$$

where

$$A = (1+\delta)/(\lambda-\delta), \quad B = (1+\lambda)/(\lambda-\delta) = A+1, \tag{C 6a, b}$$

and

$$\hat{\delta} \equiv 2\delta\lambda/(\lambda+\delta). \tag{C 7}$$

Combining (C 1)–(C 5) in (4.5), invoking (4.8) and (4.9), and expanding in ϵ, we obtain

$$\beta = 2\kappa^2[1 + (\surd 3+1)(1-\mathscr{E}_\delta)L_{|\delta|} - (\surd 3 - 1)(L_{|\delta|} + 4\ln 2) - 4(\mathscr{E}_\delta - \mathscr{E}_{2\delta})] - 4\delta_i L_o^2 + O(\epsilon) \tag{C 8a}$$

$$= 0.88 L_o - 0.76 + O(\epsilon). \tag{C 8b}$$

The contribution of J to (C 8b) is -0.14.

The integrals I, J and M (4.6) may be evaluated exactly for $D = 1$ (the mixing-length model) and the trial function (4.7) to obtain

$$\begin{aligned}\alpha + i\beta = {} & 2i\kappa^2[-1 + \tfrac{1}{4}A^2(L_\lambda + 1) + \tfrac{1}{4}B^2(L_\delta + 1) - 2AB\delta\lambda(\lambda+\delta)^{-2}(L_{\hat{\delta}} + 1)] \\ & -[\tfrac{1}{2}\lambda(1+\lambda^2)A^2(L_\lambda^2 + \tfrac{1}{6}\pi^2) + \tfrac{1}{2}\delta(1+\delta^2)B^2(L_\delta^2 + \tfrac{1}{6}\pi^2) \\ & - AB\hat{\delta}(1+\delta\lambda)(L_{\hat{\delta}}^2 + \tfrac{1}{6}\pi^2)].\end{aligned} \tag{C 9}$$

Letting $\delta \to 0$ and invoking (4.10), the numerical solution of which yields $\delta_r/\epsilon \approx 0.22$ and $\delta_i/\epsilon = -0.29$ for $0.5 < \hat{c} < 5$ and $C = 2.3 \times 10^{-3}$, we obtain

$$\begin{aligned}\beta = {} & (\kappa/\lambda)^2[\tfrac{1}{2}(L_\lambda + 1) - (1-\lambda^2)L_{|\delta|} - 4(1+\lambda)\ln 2 + \tfrac{1}{2}(1-\lambda)(1+3\lambda)] \\ & -(\delta_i/\lambda^2)(1+\lambda)(1+\lambda^2)L_\lambda^2 + O(\delta)\end{aligned} \tag{C 10a}$$

$$= 1.24 L_o + 0.03 + O(\epsilon). \tag{C 10b}$$

The contribution of J to (C 10b) is -0.32.

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